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TWO-DIMENSIONAL SUBSONIC FLOW PAST ELLIPTIC CYLINDER BY THE VARIATIONAL METHOD

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SUMMARY

A method of solution is presented for compressible fluid flow past an elliptic cylinder by means of the variational method. The solution is obtained as a function of thickness ratio and free-stream Mach number. Numerical examples are carried out for several thickness ratios and Mach numbers and the results are compared with those obtained by other methods. It is seen that the variational method yields good results for flow past a thick body at a low Mach number as well as for flow past a thin body at a high Mach number.

INTRODUCTION

In recent years the progress made in theory and practice of high-speed aerodynamics has made all the more important the fact that one should try to solve the equations of motion more accurately. Owing to the difficulty of the nonlinearity of the equations, this problem has been attacked always by approximations.

For slender bodies there is the solution given in terms of the thickness parameter, usually referred to as the Prandtl-Glauert method. On the other hand the method of Rayleigh-Janzen gives the solution in powers of Mach number and can be used in the case of thick bodies. This method was applied by Hooker (reference 1) to the study of compressible fluid flow past elliptic cylinders. But owing to the necessity of expanding a certain function in the analysis, Hooker's method cannot suitably be applied for small fineness ratios. Kaplan by using Poggi's method has investigated this problem (reference 2). Although his result is expressed in finite terms, some infinite series must be used in the course of the method. Moreover, Kaplan's solution deals with the velocity and pressure distributions on the surface of the cylinder only and not in the interior of the domain. Imai and Aihara (reference 3) also used the Rayleigh-Janzen method and obtained the flow past an elliptic cylinder in a more general manner. The use of the Rayleigh-Janzen method would involve formidable computation beyond the second approximation. For this reason this method is not very suitable for flows at high Mach numbers. Perl (reference 4) used the method of expressing the equations of motion in terms of the streamline curvature and then integrating them.

In this paper is presented a solution for compressible fluid flow past an elliptic cylinder by the direct method following the calculus of variations. The formulation of the variational principle in hydrodynamics was first due to Hargreaves (reference 5) who has shown that the integral to be used in the variational principle is a linear function of pressure. For steady irrotational flows, Bateman (reference 6) has shown that the pressure energy is an extremum. He gives the integrand as a function of the density and the velocity potential. In an earlier paper (reference 7) Bateman gives the integrand as a function of the velocity and local velocity of sound. Bateman's original integral, however, can not be directly used for fluid which occupies an infinite domain. Braun (reference 8) used the same integral without modification and attempted to obtain an approximate solution for uniform steady flow past a circular cylinder by the use of Rayleigh-Ritz method and his method leads to erroneous results. Wang (reference 9) modified the integral for such cases and it is fundamentally this method that was used by the author in this report for the study of flow past elliptic cylinders.

An elliptic coordinate system is used and in the set of simultaneous equations given in the appendix the coefficients of the terms contain as parameters the thickness ratio of the ellipse and the free-stream Mach number. Thus a method of solution for the flow past an elliptic cylinder of any fineness ratio is indicated. The results are compared with those obtained by Hooker, Kaplan, and Perl.

The present work, carried out at the Daniel Guggenheim School of Aeronautics, College of Engineering, New York University, is a major part of a thesis submitted to the Graduate Division for the partial fulfillment of the degree of Doctor of Engineering Science and has been made available to the NACA for publication because of its general interest.

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LIST OF SYMBOLS

a	velocity of sound
C_p	pressure coefficient
$G = q_m^2 - q^2$	
I	variational integral
M	Mach number (q/a)
p	pressure
q	local velocity
q_m	maximum velocity
U	velocity of uniform parallel flow at infinity
u	velocity component in x-direction
v	velocity component in y-direction
x,y	rectangular coordinates
$z = x + iy = c \cosh (\xi + i\eta)$	
γ	ratio of specific heats, 1.405 for air
η	eccentric angle
ξ,η	elliptic coordinates
ρ	density
ϕ	velocity potential
$\delta\phi$	admissible variation of ϕ

Subscripts:

i	incompressible flow

o conditions at infinity

x,y,ξ,η respective partial derivatives

EQUATIONS OF MOTION FOR COMPRESSIBLE FLUID FLOW

The various equations to be satisfied for steady, irrotational flow of a compressible fluid are given below. The continuity equation in two dimensions is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

For the flow to be irrotational one must satisfy another relation

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{2}$$

For isentropic changes of state, the equation of state is

$$p = \text{Constant} \times \rho^{\gamma}$$

The Euler equations of motion in the integrated form will then be

$$a^2 + \frac{\gamma - 1}{2} q^2 = a_o^2 + \frac{\gamma - 1}{2} q_o^2 \tag{3}$$

where γ is the ratio of specific heats and a is the velocity of sound. The subscript o refers to the conditions at infinity. Since in the variational principle, as explained later, a value of $\gamma = 2$ will be used, the above equation of state is to be replaced by

$$p = A + B\rho^{\gamma}$$

where A and B are constants. The significance of such an equation and the justification of using $\gamma = 2$ are given in reference 9. This equation of state in the differential form is

$$\frac{dp}{d\rho} = B\gamma\rho^{\gamma-1} = a^2 \tag{4}$$

One substitutes equation (4) into equation (3) and introduces

$$u = \frac{\partial\phi}{\partial x}$$

$$v = \frac{\partial\phi}{\partial y}$$

Thus $q^2 = \phi_x{}^2 + \phi_y{}^2$. By this introduction of ϕ, the irrotationality condition (equation (3)) is satisfied a priori. Equation (3) will now take the form

$$B\gamma\rho^{\gamma-1} + \frac{\gamma - 1}{2}\left(\phi_x{}^2 + \phi_y{}^2\right) = a_o{}^2 + \frac{\gamma - 1}{2} q_o{}^2$$

Since the right-hand side of this relation is a constant, it can be replaced by $\frac{\gamma - 1}{2} q_m{}^2$ where $q_m{}^2$ is dependent only upon the conditions at infinity. Then

$$B\gamma\rho^{\gamma-1} = \frac{\gamma - 1}{2}\left(q_m{}^2 - \phi_x{}^2 - \phi_y{}^2\right) \tag{5}$$

or

$$\rho = \left(\frac{\gamma - 1}{2B\gamma}\right)^{\frac{1}{\gamma-1}}\left(q_m{}^2 - \phi_x{}^2 - \phi_y{}^2\right)^{\frac{1}{\gamma-1}}$$

Substituting this value of ρ into the equation of continuity (1), one obtains the differential equation for the velocity potential

$$\frac{\partial}{\partial x}\left[\left(q_m{}^2 - \phi_x{}^2 - \phi_y{}^2\right)^{\frac{1}{\gamma-1}}\phi_x\right] + \frac{\partial}{\partial y}\left[\left(q_m{}^2 - \phi_x{}^2 - \phi_y{}^2\right)^{\frac{1}{\gamma-1}}\phi_y\right] = 0 \tag{6}$$

The boundary conditions on the velocity potential are as follows: When the flow is subsonic, the disturbances caused by the body would vanish at infinity. Thus the velocity at infinity would be that of the uniform parallel flow. The boundary of the body would be a streamline and so the normal derivative of ϕ is zero on the body. Expressed mathematically, the boundary conditions are

$$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &= U \quad \text{at infinity} \\ \frac{\partial\phi}{\partial n} &= 0 \quad \text{on the body} \end{aligned}\right\} \tag{7}$$

VARIATIONAL PRINCIPLE

The formulation of a variational principle for the motion of a nonviscous compressible fluid is given by Bateman (reference 7). He indicates the associated integral in the calculus of variations is of the type

$$I = \iint G(q)\ dx\ dy$$

Further, he suggests that for adiabatic conditions the appropriate form of G is

$$G(q) = \left[2a_o^2 + (\gamma - 1)\left(q_o^2 - q^2\right)\right]^{\frac{\gamma}{\gamma-1}} \tag{8}$$

where q_o is the velocity at infinity. Replacing $a_o^2 + \frac{\gamma - 1}{2} q_o^2$ by $\frac{\gamma - 1}{2} q_m^2$ results in

$$G(q) = (\gamma - 1)^{\frac{\gamma}{\gamma-1}}\left(q_m^2 - q^2\right)^{\frac{\gamma}{\gamma-1}}$$

Since the constant coefficient is immaterial in a variational principle, Bateman's integral can be written as

$$I = \iint \left(q_m^2 - q^2\right)^{\frac{\gamma}{\gamma-1}} dx\, dy \tag{9}$$

A proof that the integral, equation (9), gives equation (6) as its Euler equation can be found in reference 9.

For irrotational flow q^2 may be replaced by $\phi_x^2 + \phi_y^2$; then

$$I = \iint \left(q_m^2 - \phi_x^2 - \phi_y^2\right)^{\frac{\gamma}{\gamma-1}} dx\, dy \tag{9a}$$

Now ϕ is varied so that the variation $\delta\phi$ is zero on the boundary which is not a streamline; $\delta I = 0$ becomes

$$-\frac{2\gamma}{\gamma - 1} \iint \left(q_m^2 - \phi_x^2 - \phi_y^2\right)^{\frac{1}{\gamma-1}} \left(\phi_x \delta\phi_x + \phi_y \delta\phi_y\right) dx\, dy = 0 \tag{10}$$

Equation (10) can be modified as follows:

$$\begin{aligned} -\frac{2\gamma}{\gamma - 1} \iint &\left\{\frac{\partial}{\partial x}\left[\left(q_m^2 - \phi_x^2 - \phi_y^2\right)^{\frac{1}{\gamma-1}} \phi_x \delta\phi\right] + \right. \\ &\left. \frac{\partial}{\partial y}\left[\left(q_m^2 - \phi_x^2 - \phi_y^2\right)^{\frac{1}{\gamma-1}} \phi_y \delta\phi\right]\right\} dx\, dy + \\ \frac{2\gamma}{\gamma - 1} \iint \delta\phi &\left\{\frac{\partial}{\partial x}\left[\left(q_m^2 - \phi_x^2 - \phi_y^2\right)^{\frac{1}{\gamma-1}} \phi_x\right] + \right. \\ &\left. \frac{\partial}{\partial y}\left[\left(q_m^2 - \phi_x^2 - \phi_y^2\right)^{\frac{1}{\gamma-1}} \phi_y\right]\right\} dx\, dy = 0 \end{aligned} \tag{11}$$

With the aid of the divergence theorem, the first integral can be transformed into a line integral taken on the boundary. This is

$$-\frac{2\gamma}{\gamma - 1} \oint \left(q_m{}^2 - \phi_x{}^2 - \phi_y{}^2\right)^{\frac{1}{\gamma-1}} \delta\phi \frac{\partial\phi}{\partial n} \, ds \qquad (12)$$

The line integral taken on the body contour will be zero since $\frac{\partial\phi}{\partial n} = 0$ from boundary condition (7). The variation $\delta\phi$ must be so chosen as to approach zero on the other boundary. For flow in an infinite domain a correction must be made so as to insure the vanishing of the line integral as was carried out in reference 9. Although $\delta\phi$ approaches zero at infinity the line integral taken on the infinite boundary may be of finite value. This should be subtracted from integral (9), so that from equation (11) the following expression remains

$$\delta\phi \iint \left\{\frac{\partial}{\partial x}\left[\left(q_m{}^2 - \phi_x{}^2 - \phi_y{}^2\right)^{\frac{1}{\gamma-1}} \phi_x\right] + \frac{\partial}{\partial y}\left[\left(q_m{}^2 - \phi_x{}^2 - \phi_y{}^2\right)^{\frac{1}{\gamma-1}} \phi_y\right]\right\} dx\, dy = 0 \qquad (13)$$

Since $\delta\phi$ is any admissible variation, it follows that to satisfy this relation equation (6) must be satisfied in the whole domain.

ELLIPTIC COORDINATES

For solving two-dimensional flow past elliptic cylinders, it is convenient to use elliptic coordinates and transform the integral, equation (9), into the elliptic coordinate system.

Take ξ and η defined by the equation

$$z = x + iy = c \cosh (\xi + i\eta)$$

where c is a real constant.

The above relation after separation of real and imaginary parts would give

$$x = c \cosh \xi \cos \eta$$

$$y = c \sinh \xi \sin \eta$$

One takes all values from zero to infinity for ξ and values from zero to 2π for η. The curves $\xi = \text{Constant}$ and $\eta = \text{Constant}$ will be the directions of the coordinate axes and are confocal ellipses and hyperbolas, respectively. These are shown in figure 1. The angle η is usually called the "eccentric angle."

If a and b are the semimajor and semiminor axes of the ellipse given by $\xi = \xi_o$, then

$$a = c \cosh \xi_o$$

$$b = c \sinh \xi_o$$

It follows that $a^2 - b^2 = c^2$, and

$$\frac{a + b}{a - b} = e^{2\xi_o}$$

The ratio b/a is the thickness ratio of the ellipse and for simplicity the function $e^{2\xi_o} = \frac{a + b}{a - b}$ is termed t in the author's discussion and computation.

The modulus of transformation is given by

$$\left|\frac{d(\xi + i\eta)}{dz}\right| = \frac{1}{c\left(\cosh^2\xi - \cos^2\eta\right)^{1/2}} \tag{14}$$

If ϕ is the velocity potential in elliptic coordinates, the velocity components normal and tangential to an ellipse are

$$-\frac{\phi_\xi}{c\left(\cosh^2\xi - \cos^2\eta\right)^{1/2}} \quad \text{and} \quad -\frac{\phi_\eta}{c\left(\cosh^2\xi - \cos^2\eta\right)^{1/2}},$$

respectively. Hence

$$q^2 = \frac{{\phi_\xi}^2 + {\phi_\eta}^2}{c^2\left(\cosh^2\xi - \cos^2\eta\right)} \tag{15}$$

Also the elemental area in the elliptic coordinate system is

$$da = \frac{d\xi\ d\eta}{c^2\left(\cosh^2\xi - \cos^2\eta\right)} \tag{16}$$

APPLICATION OF RAYLEIGH-RITZ METHOD

After transforming the boundary-value problem into a variational problem, the Rayleigh-Ritz method can be used to obtain close approximations to the solution. Using relations obtained in the previous section, variational integral (9) in the elliptic coordinate system would be, with a value of 2 for γ,

$$I = \iint \left[{q_m}^2 - \frac{{\phi_\xi}^2 + {\phi_\eta}^2}{c^2\left(\cosh^2\xi - \cos^2\eta\right)}\right]^2 c^2\left(\cosh^2\xi - \cos^2\eta\right)\ d\xi\ d\eta \tag{17}$$

and it is proved in reference 6 that for subsonic flow the integral I is a maximum for all admissible variations of ϕ.

The Rayleigh-Ritz method is a direct method and consists of approximating ϕ by a finite sequence of functions. For ϕ in the integral (equation (17)), substitute

$$\phi = \sum_{k=1}^{n} a_k\phi_k \tag{18}$$

with ϕ_k satisfying the boundary conditions and with a_k as undetermined parameters.

Since the required solution for ϕ makes the integral a maximum, one may transform the variational problem into an ordinary maximizing problem. Thus one can obtain a system of simultaneous equations by taking partial derivatives with respect to the various constants:

$$\frac{\partial I}{\partial a_1} = 0, \qquad \frac{\partial I}{\partial a_2} = 0, \qquad \frac{\partial I}{\partial a_3} = 0$$

and so forth. Solving these simultaneous equations, the constants can be obtained. Then

$$\phi = a_1\phi_1 + a_2\phi_2 + a_3\phi_3 \ . \ . \ . \tag{18a}$$

makes the integral (equation (17)) stationary and is a solution of the problem.

In the case where more than one set of values for these constants is obtained the solution which makes the integral a maximum is to be taken as the required solution.

EXTENSION TO INFINITE DOMAIN

In applying the variational integral, equation (9), to the present case when the domain extends to infinity, the value of the integral becomes infinite. But this does not prevent one from obtaining a solution by maximizing or minimizing such an integral. The difficulty would arise only in the case when this integral has to be evaluated.

There is another feature in connection with line integral (12) which should be scrutinized carefully since it is taken around an infinite boundary. This should be zero to satisfy the conditions upon which the variational problem is formulated. Even though $\delta\phi$ approaches zero at infinity, the product $\delta\phi$ ds may contain certain terms of zero power in r. Then line integral (12) will have a finite value and will have to be accounted for by subtracting it from the integral of equation (9).

Consider line integral (12) around an ellipse given by a large value of ξ and then let ξ approach infinity. One obtains for

$$q^2 = \frac{\phi_\xi^{\ 2} + \phi_\eta^{\ 2}}{c^2\left(\cosh^2\xi - \cos^2\eta\right)}$$

with ϕ assumed as given in a later section, equation (23), at large distances

$$q^2 \approx U^2 + 0\left(\frac{1}{\cosh^2\xi}\right)$$

where $0\left(\frac{1}{\cosh^2\xi}\right)$ means terms approaching zero to the order of $1/\cosh^2\xi$ and higher as $\xi \longrightarrow \infty$.

In the line integral, $\frac{\partial\phi}{\partial n}$ is in the direction of the inward normal. Therefore

$$\frac{\partial\phi}{\partial n} = h\,\frac{\partial\phi}{\partial\xi}$$

and the elemental length

$$ds = \frac{1}{h}\,d\eta$$

where h is the modulus of transformation. Hence

$$\frac{\partial\phi}{\partial n}\,ds = \frac{\partial\phi}{\partial\xi}\,d\eta$$

and with ϕ as in equation (23), at large values of ξ,

$$\frac{\partial\phi}{\partial n}\,ds = \left[U(a + b)\,\sinh\,\xi\,\cos\,\eta + 0\left(\frac{1}{\cosh^2\xi}\right)\right]d\eta \tag{19}$$

In equation (23), the assumption is that the flow at infinity is prescribed and is equal to the incompressible flow. Hence, the variation $\delta\phi$ for large values of ξ will be the same as the difference between the assumed ϕ and the incompressible velocity potential. Thus

$$\delta\phi = A_{11}\frac{\cos\eta}{\cosh\xi} + A_{13}\frac{\cos^3\eta}{\cosh\xi} + 0\left(\frac{1}{\cosh^3\xi}\right) \tag{20}$$

Substituting these into line integral (12)

$$-\frac{2\gamma}{\gamma-1}\oint\left(q_m{}^2 - q^2\right)^{\frac{1}{\gamma-1}}\delta\phi\,\frac{\partial\phi}{\partial n}\,ds =$$

$$-4\int_0^{2\pi}\left(q_m{}^2 - U^2\right)U(a+b)\ \sinh\xi\ \cos\eta\left(A_{11}\frac{\cos\eta}{\cosh\xi} + \right.$$

$$\left.A_{13}\frac{\cos^3\eta}{\cosh\xi}\right)d\eta + 0\left(\frac{1}{\cosh^2\xi}\right)$$

After integrating and letting $\xi \longrightarrow \infty$, the above expression becomes

$$-4\left(q_m{}^2 - U^2\right)U(a+b)\left(\pi A_{11} + \frac{3}{4}\pi A_{13}\right) \tag{21}$$

This is the term which should be subtracted from the integral of equation (17).

FLOW PAST AN ELLIPTIC CYLINDER

The velocity potential for incompressible flow past an elliptic cylinder defined by $\xi = \xi_o$ is

$$\phi_i = U(a+b)\ \cosh\left(\xi - \xi_o\right)\ \cos\eta \tag{22}$$

where a and b are semiaxes of the ellipse. This can be readily checked to satisfy Laplace's equation in elliptic coordinates

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} = 0$$

To investigate the two-dimensional flow of a compressible fluid past an elliptic cylinder given by $\xi = \xi_o$, one may assume ϕ in the following form:

$$\phi = \phi_i + A_{11} \frac{\cos \eta}{\cosh (\xi - \xi_o)} + A_{13} \frac{\cos^3\eta}{\cosh (\xi - \xi_o)} + A_{31} \frac{\cos \eta}{\cosh^3(\xi - \xi_o)} + A_{33} \frac{\cos^3\eta}{\cosh^3(\xi - \xi_o)} + \ldots \tag{23}$$

where ϕ_i is the potential for incompressible flow and A_{11}, A_{13}, A_{31}, and A_{33} are undetermined parameters.

In the above assumed potential (equation (23)) all the additional terms approach zero at infinity to the order of $1/\cosh \xi$. This leaves only the potential of the incompressible flow which is the boundary condition at infinity. Also it can be seen that for the additional terms the derivatives with respect to ξ vanish at $\xi = \xi_o$. Thus the normal velocity component on the body due to the additional terms is zero. The assumed variation (equation (23)) therefore satisfies the required boundary conditions. If the flow were not symmetrical, one would have to add to expression (23) terms containing $\sin \eta$, $\sin^3\eta$, and so forth.

The assumed ϕ function (equation (23)) is now substituted into the variational integral, equation (9). To cover the whole domain exterior to the ellipse given by $\xi = \xi_o$, the limits for integration are from 0 to 2π for η and from ξ_o to ∞ for ξ. By differentiating under the integral sign, one obtains the following four simultaneous equations:

$$\left.\begin{aligned} \frac{\partial I}{\partial A_{11}} = 0 \qquad \frac{\partial I}{\partial A_{13}} = 0 \\ \frac{\partial I}{\partial A_{31}} = 0 \qquad \frac{\partial I}{\partial A_{33}} = 0 \end{aligned}\right\} \tag{24}$$

Integrating these equations, four algebraic simultaneous equations in the parameters A_{11}, A_{13}, A_{31}, and A_{33} are finally obtained and are given in appendix A. The equations given in appendix A are divided into two parts. The linear terms in the parameters are on the left-hand side of the equations. All the nonlinear terms are grouped together and called K_1, K_2, K_3, and K_4. These are given at the right-hand side of the equations, together with the constant terms.

NUMERICAL EXAMPLE

The flow past an elliptic cylinder of thickness ratio $b/a = 0.5$ when the Mach number of uniform flow at infinity is 0.5 is investigated and the method used in solving the equations is illustrated here.

The value of the major axis of the ellipse is taken as a unit of dimension, with $b/a = 0.5$. Then

$$t = \frac{a + b}{a - b} = 3$$

Divide throughout the equations of appendix A by the velocity of sound at infinity, so as to make the coefficients nondimensional. Substituting these values of U/a_o, q_m^2, and t in the four simultaneous equations of appendix A and evaluating all the coefficients, one obtains the following system of equations:

$$1.99301954A_{11}' + 1.67206436A_{13}' + 1.62409948A_{31}' + 1.32770102A_{33}' =$$

$$0.09169413 + \frac{1}{2.25} K_1'$$

$$1.67206426A_{11}' + 2.28737268A_{13}' + 1.3832566A_{31}' + 1.71470758A_{33}' =$$

$$0.03621830 + \frac{1}{2.25} K_2'$$

$$1.62409948A_{11}' + 1.38325662A_{13}' + 1.96462204A_{31}' + 1.59203403A_{33}' =$$

$$0.07905474 + \frac{1}{2.25} K_3'$$

$$1.32770102A_{11}' + 1.71470758A_{13}' + 1.59203403A_{31}' + 1.78968390A_{33}' =$$

$$0.03812186 + \frac{1}{2.25} K_4'$$

where K_1', K_2', K_3', and K_4' are the proper values of K_1, K_2, K_3, and K_4 of appendix A with the values of t, U/a_o, and q_m^2 substituted and A_{11}', A_{13}', and so forth in the above set of equations are the assumed coefficients of appendix A, each divided by the velocity of sound at infinity.

Now one may make use of Crout's method (reference 10) and solve the above set of equations in terms of the right-hand-side members of each equation. Since the above matrix is symmetrical, it is quite easy to obtain the following expressions:

$$A_{11}' = 4.67065623\lambda_1 - 3.95643729\lambda_2 - 4.79816972\lambda_3 + 4.5396629\lambda_4$$

$$A_{13}' = -3.95643629\lambda_1 + 4.99700942\lambda_2 + 4.49078519\lambda_3 - 5.84736055\lambda_4$$

$$A_{31}' = -4.79816982\lambda_1 + 4.49078533\lambda_2 + 6.86307052\lambda_3 - 6.84818730\lambda_4$$

$$A_{33}' = 4.59396635\lambda_1 - 5.84736064\lambda_2 - 6.84818723\lambda_3 + 8.84494015\lambda_4$$

where

$$\lambda_1 = 0.09169413 + \frac{1}{2.25} K_1'$$

$$\lambda_2 = 0.03621830 + \frac{1}{2.25} K_2'$$

$$\lambda_3 = 0.07905474 + \frac{1}{2.25} K_3'$$

$$\lambda_4 = 0.03812186 + \frac{1}{2.25} K_4'$$

For the computation of A_{ij}', first one puts all the nonlinear terms as zero and obtains the values of the coefficients. Next these are substituted in the complete equations and the values of the coefficients are again solved for. This procedure is repeated and the above set of equations is solved by successive iterations, until the coefficients iterate themselves within a desired accuracy. The following are the values obtained:

$$\left.\begin{aligned} A_{11}' &= 0.08335840 \\ A_{13}' &= -0.05016832 \\ A_{31}' &= 0.00714746 \\ A_{33}' &= 0.00157309 \end{aligned}\right\} \tag{25}$$

These values are substituted in the expression for the velocity potential, equation (23). The velocity at a point defined by the eccentric angle η on the elliptic cylinder is given by

$$q = \frac{1}{c\left(\cosh^2\xi_o - \cos^2\eta\right)^{1/2}} \phi_\eta$$

The value of ξ_o is obtained from

$$t = e^{2\xi_o} = 3$$

With the value of ϕ substituted in the above equation and after dividing through by the velocity U at infinity, q/U becomes

$$\frac{q}{U} = \frac{1}{c\left(\cosh^2\xi_o - \cos^2\eta\right)^{1/2}}\left(1 + \frac{b}{a}\right) \sin\eta + \frac{A_{11}'}{M_o} \sin\eta +$$

$$\frac{3A_{13}'}{M_o} \cos^2\eta \sin\eta + \frac{A_{31}'}{M_o} \sin\eta + \frac{3A_{33}'}{M_o} \cos^2\eta \sin\eta \tag{26}$$

With the obtained values (equations (25)) of the coefficients, $b/a = 0.5$, and $M_o = 0.5$, the above equation would be

$$\frac{q}{U} = \frac{1.15470108}{\left(\frac{4}{3} - \cos^2\eta\right)^{1/2}}\left(1.68101172 - 0.29157138 \cos^2\eta\right) \sin \eta \qquad (27)$$

These values of q/U are computed for various angles η and are given in table 1. The results are also shown plotted in figure 2.

From the pressure-density relation for adiabatic flow one can obtain (reference 11) the following relation between the local pressure and pressure at infinity:

$$\frac{p - p_o}{\frac{1}{2}\rho_o U^2} = \frac{1}{\frac{\gamma}{2} M_o^2}\left(\left\{1 + \frac{\gamma - 1}{2} M_o^2 \left[1 - \left(\frac{q}{U}\right)^2\right]\right\}^{\frac{\gamma}{\gamma-1}} - 1\right) \qquad (28)$$

Note that $\gamma = 2$ was used in the process of obtaining a solution for ϕ. It is possible to use $\gamma = 1.5$ or $\gamma = 1.33$, but it is seen that the change in γ does not alter the computed values of the velocities appreciably.

Even though $\gamma = 2$ is used for getting the velocities it is proposed to use $\gamma = 1.405$ in computing pressures. Then

$$C_p = \frac{p - p_o}{\frac{1}{2}\rho_o U^2} = \frac{1}{0.7025M_o^2}\left(\left\{1 + 0.2025M_o^2 \left[1 - \left(\frac{q}{U}\right)^2\right]\right\}^{\frac{1.405}{0.405}} - 1\right) \qquad (29)$$

The pressure computed as above is shown in figure 3, as a pressure coefficient, against η and is compared with the pressure distributions obtained by the Prandtl-Glauert method and the Kármán-Tsien method (reference 12).

By substituting $t = 11/9$ in equation (24), the flow past an elliptic cylinder of thickness ratio 0.1 is computed when $M_o = 0.8$, and the results are shown in figures 4 and 5.

Using Poggi's method Kaplan (reference 2) has calculated the flow past elliptic cylinders of thickness ratios 0.5 and 0.1. These results are shown in tables 1 and 2 and are plotted in figures 2 and 4 for comparison.

By letting t approach infinity the flow past a circular cylinder is obtained and the result is shown in table 3 and figure 6. Imai (reference 13) has calculated the subsonic flow past a circular cylinder to the sixth power of the stream Mach number. These results are shown in table 3 and figure 6. The results obtained by Wang (reference 9) show a higher velocity throughout. This is due to the fact that Wang used six terms in the series assumed for the velocity potential. In the present investigation only four terms of the series are used and hence lower values for the velocities are obtained.

CONCLUDING DISCUSSION

The velocities obtained by the variational method show more deviation from the incompressible flow than indicated by Kaplan, Hooker, and other investigators. From the velocity distribution shown in figures 2 and 4, it is seen that the variational method yields a good result for flow past a thick body at a low Mach number, as well as for flow past a thin body at a high Mach number. There is close agreement with Perl's results at both thickness ratios.

The results obtained by the variational method for the flow past an elliptic cylinder of 0.1 thickness ratio at a Mach number $M_0 = 0.8$ are given in table 2. The maximum velocity occurs at the ends of the minor axis and is given as $q/U = 1.2255$. With the aid of the well-known relation between velocity and the stream density for adiabatic flow, the maximum stream density is computed for the above velocity. A maximum stream density of 0.575 is obtained by such computation. Using the Prandtl-Busemann method, the second approximation carried out by Schmieden and Kawalki (reference 14) shows a maximum stream density of 0.583. Further, Schmieden and Kawalki indicate that the value of stream density should naturally coincide with 0.578 given by the sonic boundary. The present method gives results higher than those of Kaplan (reference 2) and more in agreement with references 4 and 14.

For the flow past a circular cylinder, the pressure coefficients obtained by the present method are between those of the Prandtl-Glauert and Kármán-Tsien methods.

For flow past elliptic cylinders the pressure coefficients obtained at $\eta = \pi/2$ are higher than those from the Kármán-Tsien method also.

From figures 3 and 5 it appears that the local Mach number has more effect than that indicated by the Kármán-Tsien method. In figure 7 the flow past a cylinder of 0.1 thickness ratio is shown at various Mach numbers. At higher Mach numbers the variational method gives results which differ considerably from those obtained by other methods. This deviation appears in Hooker's results to some extent, as shown in figure 2. This may be due to the fact that the compressibility effect is far higher at points where the local velocity is nearer the critical value. The methods of Prandtl-Glauert and Kármán-Tsien for obtaining the compressibility effect are independent of the critical Mach number of the flow. But in the variational method the critical Mach number directly affects the coefficients in the velocity potential. In fact, at some value higher than the critical Mach number the coefficients do not converge on successive iterations.

The point of zero pressure on the surface of a cylinder moves rearward when compressibility is taken into account. This is not the case when the Prandtl-Glauert and Kármán-Tsien methods are used. The investigations of Kaplan and Hooker, however, show this deviation which is consistent with the experimental results.

New York University
New York, N. Y., August 1, 1949

APPENDIX A

EQUATIONS FOR FLOW PAST ELLIPTIC CYLINDER OF THICKNESS RATIO $\frac{t-1}{t+1}$ AT $M_o = U/a_o$

The equations for the flow past an elliptic cylinder of thickness ratio $\frac{t-1}{t+1}$ at $M_o = U/a_o$ are as follows, where A, B, C, D, and so forth are certain definite integrals containing the parameter t and are defined at the end of the appendix.

First equation:

$$A_{11}\left[1.333333q_m^2 - U^2(6B + 6tK - 1.18070817)\right] +$$

$$A_{13}\left[1.0q_m^2 - U^2\left(6B + 2.47817466 - 0.59035447\,\frac{1}{t}\right)\right] +$$

$$A_{31}\left[1.06666667q_m^2 - U^2(72tC + 24tL - 1.54737540)\right] +$$

$$A_{33}\left[0.8q_m^2 - U^2\left(72tC + 1.31446908 - 0.54340470\,\frac{1}{t}\right)\right] =$$

$$U^3\,\frac{2at}{t+1}\left(-\frac{A}{2} + \frac{J}{2}\,t + 1.59111692 + \frac{1}{2}\log_e\frac{t}{t+1}\right) + \left(\frac{t+1}{2at}\right)^2 K_1$$

Second equation:

$$A_{11}\left[q_m^2 - U^2\left(6B + 2.47817466 - 0.59035447\,\frac{1}{t}\right)\right] +$$

$$A_{13}\left[1.33333333q_m^2 - U^2\left(6B + 2.85882167 - 0.02182589\,\frac{1}{t} - 0.28250893\,\frac{1}{t^2}\right)\right] +$$

$$A_{31}\left[0.8q_m^2 - U^2\left(72tC + 1.10262516 - 0.57454014\,\frac{1}{t}\right)\right] +$$

$$A_{33}\left[q_m^2 - U^2\left(72tC + 1.43683614 - 0.06345864\,\frac{1}{t} - 0.27337794\,\frac{1}{t^2}\right)\right] =$$

$$U^3\,\frac{2at}{t+1}\left(-\frac{A}{2} + 1.78426410 - 0.19314718\,\frac{1}{t}\right) + \left(\frac{t+1}{2at}\right)^2 K_2$$

Third equation:

$$A_{11}\left[1.06666667q_m{}^2 - U^2(72tC + 24tL - 1.54737540)\right] +$$

$$A_{13}\left[0.8q_m{}^2 - U^2\left(72tC + 1.10262516 - 0.57454014\ \frac{1}{t}\right)\right] +$$

$$A_{31}\left[1.21904693q_m{}^2 - U^2(864D + 96tM - 0.29524866)\right] +$$

$$A_{33}\left[0.91428519q_m{}^2 - U^2\left(864D + 1.94284848 - 0.52783764\ \frac{1}{t}\right)\right] =$$

$$U^3\ \frac{2at}{t + 1}\left(-6B + 2tK - 1.27411439\right) + \left(\frac{t + 1}{2at}\right)^2 K_3$$

Fourth equation:

$$A_{11}\left[0.8q_m{}^2 - U^2\left(72tC + 1.31446908 - 0.54340470\ \frac{1}{t}\right)\right] +$$

$$A_{13}\left[q_m{}^2 - U^2\left(72tC + 1.43683614 - 0.06345864\ \frac{1}{t} - 0.27337794\ \frac{1}{t^2}\right)\right] +$$

$$A_{31}\left[0.91438519q_m{}^2 - U^2\left(864D + 1.94284848 - 0.52783764\ \frac{1}{t}\right)\right] +$$

$$A_{33}\left[1.02857103q_m{}^2 - U^2\left(864D + 2.06249040 - 0.08461188\ \frac{1}{t} - 0.26469675\ \frac{1}{t^2}\right)\right] =$$

$$U^3\ \frac{2at}{t + 1}\left(-6B + 0.29441406 - 0.18223425\ \frac{1}{t}\right) + \left(\frac{t + 1}{2at}\right)^2 K_4$$

The nonlinear terms K_1, K_2, K_3, and K_4 are as follows:

$$
\begin{aligned}
K_1 = {} & A_{11}^3(32D + 32tM - 0.30476230) + A_{13}^3\left(32D + 0.90624977 + 0.19262107 \frac{1}{t} - 0.12594008 \frac{1}{t^2} - 0.04971839 \frac{1}{t^3}\right) + \\
& A_{31}^3\left(\frac{55296}{t^3} G + 2048tQ - 0.14097408 + 0.09345024 \frac{1}{t} + 0.03760128 \frac{1}{t^2} + 0.01714176 \frac{1}{t^3}\right) + \\
& A_{33}^3\left(\frac{55296}{t^3} G + 0.78763104 + 0.23067504 \frac{1}{t} - 0.08105616 \frac{1}{t^2} - 0.02979936 \frac{1}{t^3}\right) + \\
& A_{11}^2U \frac{2at}{t + 1}(-24tC + 24tL - 1.10000028) + A_{11}^2A_{13}\left(96D + 1.77142776 - 0.442857 \frac{1}{t}\right) + \\
& A_{11}^2A_{31}\left(\frac{1152}{t} E + 384tN - 0.46412160 + 0.05704704 \frac{1}{t}\right) + A_{11}^2A_{33}\left(\frac{1152}{t} E + 1.68942240 - 0.35366256 \frac{1}{t}\right) + \\
& A_{13}^2U \frac{2at}{t + 1}\left(-24tC + 1.92993774 - 0.02555028 \frac{1}{t} - 0.27105478 \frac{1}{t^2}\right) + \\
& A_{13}^2A_{11}\left(96D + 1.97916568 + 0.06166172 \frac{1}{t} - 0.25047667 \frac{1}{t^2}\right) + \\
& A_{13}^2A_{31}\left(\frac{1152}{t} E + 2.015150 + 0.17372192 \frac{1}{t} - 0.23433120 \frac{1}{t^2}\right) + \\
& A_{13}^2A_{33}\left(\frac{1152}{t} E + 2.72042070 + 0.69054222 \frac{1}{t} - 0.34588152 \frac{1}{t^2} - 0.14399928 \frac{1}{t^3}\right) + \\
& A_{31}^2U \frac{2at}{t + 1}\left(-\frac{3456}{t} E + 384tN - 1.16001280 - 0.17114112 \frac{1}{t}\right) + \\
& A_{31}^2A_{11}\left(\frac{13824}{t^2} F + 1536tP - 0.14124928 + 0.12966912 \frac{1}{t} + 0.04409856 \frac{1}{t^2}\right) + \\
& A_{31}^2A_{13}\left(\frac{13824}{t^2} F + 1.95553600 - 0.20380672 \frac{1}{t} + 0.04409856 \frac{1}{t^2}\right) + \\
& A_{31}^2A_{33}\left(\frac{165888}{t^3} G + 2.15970816 - 0.03670656 \frac{1}{t} + 0.11280384 \frac{1}{t^2} + 0.05142528 \frac{1}{t^3}\right) + \\
& A_{33}^2U \frac{2at}{t + 1}\left(-\frac{3456}{t} E + 1.19026800 - 0.14974848 \frac{1}{t} - 0.24776064 \frac{1}{t^2}\right) + \\
& A_{33}^2A_{11}\left(\frac{13824}{t^2} F + 1.91459592 + 0.22063104 \frac{1}{t} - 0.18427392 \frac{1}{t^2}\right) + \\
& A_{33}^2A_{13}\left(\frac{13824}{t^2} F + 2.71604124 + 0.77606208 \frac{1}{t} - 0.27906768 \frac{1}{t^2} - 0.13974606 \frac{1}{t^3}\right) + \\
& A_{33}^2A_{31}\left(\frac{165888}{t^3} G + 2.25168768 + 0.42723072 \frac{1}{t} - 0.09931680 \frac{1}{t^2} + 0.05142528 \frac{1}{t^3}\right) + \\
& A_{11}A_{13}U \frac{2at}{t + 1}\left(-48tC + 3.54737568 - 1.03333332 \frac{1}{t}\right) + A_{11}A_{31}U \frac{2at}{t + 1}(-576D + 192tM - 2.38094556) + \\
& A_{11}A_{33}U \frac{2at}{t + 1}\left(-576D + 2.57143464 - 0.94285944 \frac{1}{t}\right) + A_{13}A_{31}U \frac{2at}{t + 1}\left(-576D + 2.57143464 - 0.94285944 \frac{1}{t}\right) + \\
& A_{13}A_{33}U \frac{2at}{t + 1}\left(-576D + 2.87500632 - 0.01465512 \frac{1}{t} - 0.51822222 \frac{1}{t^2}\right) + \\
& A_{31}A_{33}U \frac{2at}{t + 1}\left(-\frac{6912}{t} E + 2.09204160 - 1.19231904 \frac{1}{t}\right) + A_{11}A_{13}A_{31}\left(\frac{2304}{t} E + 3.56932160 - 0.66923104 \frac{1}{t}\right) + \\
& A_{11}A_{13}A_{33}\left(\frac{2304}{t} E + 3.79220400 + 0.27125568 \frac{1}{t} - 0.47769408 \frac{1}{t^2}\right) + \\
& A_{11}A_{31}A_{33}\left(\frac{27648}{t^2} F + 3.61908864 - 0.47112192 \frac{1}{t} + 0.08819712 \frac{1}{t^2}\right) + \\
& A_{13}A_{31}A_{33}\left(\frac{27648}{t^2} F + 3.89762544 + 0.46507776 \frac{1}{t} - 0.36557280 \frac{1}{t^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
K_2 = {} & A_{11}^3\left(32D + 0.49047590 - 0.16538478\,\frac{1}{t}\right) + \\
& A_{13}^3\left(32D + 1.493750 + 0.53412758\,\frac{1}{t} - 0.14506687\,\frac{1}{t^2} - 0.14991404\,\frac{1}{t^3} - 0.03051809\,\frac{1}{t^4}\right) + \\
& A_{31}^3\left(\frac{55296}{t^3}\,G + 0.38134272 - 0.08972160\,\frac{1}{t} + 0.03760128\,\frac{1}{t^2} + 0.01714176\,\frac{1}{t^3}\right) + \\
& A_{33}^3\left(\frac{55296}{t^3}\,G + 1.13567832 + 0.46610640\,\frac{1}{t} - 0.11696238\,\frac{1}{t^2} - 0.12707712\,\frac{1}{t^3} - 0.02943540\,\frac{1}{t^4}\right) + \\
& A_{11}^2U\,\frac{2at}{t+1}\left(-24tC + 1.77368784 - 0.51666667\,\frac{1}{t}\right) + A_{11}^2A_{13}\left(96D + 1.97916568 + 0.06166172\,\frac{1}{t} - 0.25047667\,\frac{1}{t^2}\right) + \\
& A_{11}^2A_{31}\left(\frac{1152}{t}\,E + 1.78466080 - 0.33461552\,\frac{1}{t}\right) + A_{11}^2A_{33}\left(\frac{1152}{t}\,E + 1.896102 + 0.13562784\,\frac{1}{t} - 0.23884706\,\frac{1}{t^2}\right) + \\
& A_{13}^2U\,\frac{2at}{t+1}\left(-24tC + 2.844495 + 0.52236534\,\frac{1}{t} - 0.41226033\,\frac{1}{t^2} - 0.15460086\,\frac{1}{t^3}\right) + \\
& A_{13}^2A_{11}\left(96D + 2.84999937 + 0.62449833\,\frac{1}{t} - 0.36896406\,\frac{1}{t^2} - 0.14839629\,\frac{1}{t^3}\right) + \\
& A_{13}^2A_{31}\left(\frac{1152}{t}\,E + 2.84542110 + 0.74054070\,\frac{1}{t} - 0.33572088\,\frac{1}{t^2} - 0.14308344\,\frac{1}{t^3}\right) + \\
& A_{13}^2A_{33}\left(\frac{1152}{t}\,E + 4.49892738 + 1.75572360\,\frac{1}{t} - 0.36954288\,\frac{1}{t^2} - 0.43131528\,\frac{1}{t^3}\right) + \\
& A_{31}^2U\,\frac{2at}{t+1}\left(-\frac{3456}{t}\,E + 1.14125920 - 0.57711248\,\frac{1}{t}\right) + A_{31}^2A_{11}\left(\frac{13824}{t^2}\,F + 1.955536 - 0.20380672\,\frac{1}{t} + 0.04409856\,\frac{1}{t^2}\right) + \\
& A_{31}^2A_{13}\left(\frac{13824}{t^2}\,F + 2.27957512 + 0.347648\,\frac{1}{t} - 0.16840704\,\frac{1}{t^2}\right) + \\
& A_{31}^2A_{33}\left(\frac{165888}{t^3}\,G + 2.47248768 + 0.50808576\,\frac{1}{t} - 0.08864160\,\frac{1}{t^2} + 0.05142528\,\frac{1}{t^3}\right) + \\
& A_{33}^2U\,\frac{2at}{t+1}\left(-\frac{3456}{t}\,E + 1.88874090 + 0.32836194\,\frac{1}{t} - 0.37832616\,\frac{1}{t^2} - 0.14708412\,\frac{1}{t^3}\right) + \\
& A_{33}^2A_{11}\left(\frac{13824}{t^2}\,F + 2.71604124 + 0.77606208\,\frac{1}{t} - 0.279067068\,\frac{1}{t^2} - 0.13974606\,\frac{1}{t^3}\right) + \\
& A_{33}^2A_{13}\left(\frac{13824}{t^2}\,F + 4.43507589 + 1.84144320\,\frac{1}{t} - 0.28275813\,\frac{1}{t^2} - 0.41674986\,\frac{1}{t^3} - 0.08774370\,\frac{1}{t^4}\right) + \\
& A_{33}^2A_{31}\left(\frac{165888}{t^3}\,G + 3.11637312 + 1.03692240\,\frac{1}{t} - 0.16510608\,\frac{1}{t^2} - 0.08204112\,\frac{1}{t^3}\right) + \\
& A_{11}A_{13}U\,\frac{2at}{t+1}\left(-48tC + 3.85987548 - 0.05110056\,\frac{1}{t} - 0.54210956\,\frac{1}{t^2}\right) + \\
& A_{11}A_{31}U\,\frac{2at}{t+1}\left(-576D + 2.57143464 - 0.94285944\,\frac{1}{t}\right) + \\
& A_{11}A_{33}U\,\frac{2at}{t+1}\left(-576D + 2.87500632 - 0.01465512\,\frac{1}{t} - 0.51822222\,\frac{1}{t^2}\right) + \\
& A_{13}A_{31}U\,\frac{2at}{t+1}\left(-576D + 2.70833960 - 0.06203048\,\frac{1}{t} - 0.52347038\,\frac{1}{t^2}\right) + \\
& A_{13}A_{33}U\,\frac{2at}{t+1}\left(-576D + 4.37500434 + 0.97810890\,\frac{1}{t} - 0.79421988\,\frac{1}{t^2} - 0.30175410\,\frac{1}{t^3}\right) + \\
& A_{31}A_{33}U\,\frac{2at}{t+1}\left(-\frac{6912}{t}\,E + 2.38053600 - 0.29949696\,\frac{1}{t} - 0.49552128\,\frac{1}{t^2}\right) + \\
& A_{11}A_{13}A_{31}\left(\frac{2304}{t}\,E + 4.030300 + 0.34744384\,\frac{1}{t} - 0.46866240\,\frac{1}{t^2}\right) + \\
& A_{11}A_{13}A_{33}\left(\frac{2304}{t}\,E + 5.44084140 + 1.38108444\,\frac{1}{t} - 0.69176304\,\frac{1}{t^2} - 0.28799856\,\frac{1}{t^3}\right) + \\
& A_{11}A_{31}A_{33}\left(\frac{27648}{t^2}\,F + 4.10292624 + 0.53652480\,\frac{1}{t} - 0.35664768\,\frac{1}{t^2}\right) + \\
& A_{13}A_{31}A_{33}\left(\frac{27648}{t^2}\,F + 5.38417896 + 1.53128544\,\frac{1}{t} - 0.56259792\,\frac{1}{t^2} - 0.27990684\,\frac{1}{t^3}\right)
\end{aligned}
$$

$$
\begin{aligned}
K_3 = {} & A_{11}^3\left(\frac{384}{t} E + 128tN - 0.20232640 + 0.01901568 \frac{1}{t}\right) + \\
& A_{13}^3\left(\frac{384}{t} E + 0.91722360 + 0.23434728 \frac{1}{t} - 0.11444712 \frac{1}{t^2} - 0.04792344 \frac{1}{t^3}\right) + \\
& A_{31}^3\left(\frac{663552}{t^4} H + 8192tR + 0.18477056 + 0.25878528 \frac{1}{t} + 0.08626176 \frac{1}{t^2} + 0.04644864 \frac{1}{t^3} + 0.01327104 \frac{1}{t^4}\right) + \\
& A_{33}^3\left(\frac{663552}{t^4} H + 1.03949568 + 0.43053768 \frac{1}{t} - 0.015120 \frac{1}{t^2} + 0.00181440 \frac{1}{t^3} + 0.01327104 \frac{1}{t^4}\right) + \\
& A_{11}^2 U \frac{2at}{t+1}(-288D + 96tM - 1.19047278) + A_{11}^2 A_{13}\left(\frac{1152}{t} E + 1.78466080 - 0.33461552 \frac{1}{t}\right) + \\
& A_{11}^2 A_{31}\left(\frac{13824}{t^2} F + 1536tP - 0.14124928 + 0.12966912 \frac{1}{t} + 0.04409856 \frac{1}{t^2}\right) + \\
& A_{11}^2 A_{33}\left(\frac{13824}{t^2} F + 1.80954432 - 0.23556096 \frac{1}{t} + 0.04409856 \frac{1}{t^2}\right) + \\
& A_{13}^2 U \frac{2at}{t+1}\left(-288D + 1.35416980 - 0.03101524 \frac{1}{t} - 0.26173519 \frac{1}{t^2}\right) + \\
& A_{13}^2 A_{11}\left(\frac{1152}{t} E + 2.015150 + 0.17372192 \frac{1}{t} - 0.23433120 \frac{1}{t^2}\right) + \\
& A_{13}^2 A_{31}\left(\frac{13824}{t^2} F + 2.27957512 + 0.34764800 \frac{1}{t} - 0.16840704 \frac{1}{t^2}\right) + \\
& A_{13}^2 A_{33}\left(\frac{13824}{t^2} F + 2.90765532 + 0.85941696 \frac{1}{t} - 0.26121856 \frac{1}{t^2} - 0.13808882 \frac{1}{t^3}\right) + \\
& A_{31}^2 U \frac{2at}{t+1}\left(-\frac{41472}{t^2} F + 1536tP - 1.35762816 - 0.38900736 \frac{1}{t} + 0.13229568 \frac{1}{t^2}\right) + \\
& A_{31}^2 A_{11}\left(\frac{165888}{t^3} G + 6144tQ + 0.37195776 + 0.28035072 \frac{1}{t} + 0.11280384 \frac{1}{t^2} + 0.05142528 \frac{1}{t^3}\right) + \\
& A_{31}^2 A_{13}\left(\frac{165888}{t^3} G + 2.33634816 + 0.00372096 \frac{1}{t} + 0.11280384 \frac{1}{t^2} + 0.05142528 \frac{1}{t^3}\right) + \\
& A_{31}^2 A_{33}\left(\frac{1990656}{t^4} H + 2.81806848 + 0.49208832 \frac{1}{t} + 0.25878528 \frac{1}{t^2} + 0.13834592 \frac{1}{t^3} + 0.03981312 \frac{1}{t^4}\right) + \\
& A_{33}^2 U \frac{2at}{t+1}\left(-\frac{41472}{t^2} F + 0.87557832 - 0.30465792 \frac{1}{t} - 0.36683712 \frac{1}{t^2}\right) + \\
& A_{33}^2 A_{11}\left(\frac{165888}{t^3} G + 2.25168768 + 0.42723072 \frac{1}{t} - 0.09931680 \frac{1}{t^2} + 0.05142528 \frac{1}{t^3}\right) + \\
& A_{33}^2 A_{13}\left(\frac{165888}{t^3} G + 3.11637312 + 1.03692240 \frac{1}{t} - 0.16510608 \frac{1}{t^2} - 0.08204112 \frac{1}{t^3}\right) + \\
& A_{33}^2 A_{31}\left(\frac{1990656}{t^4} H + 2.91644928 + 0.97228800 \frac{1}{t} + 0.06590592 \frac{1}{t^2} + 0.13934592 \frac{1}{t^3} + 0.03981312 \frac{1}{t^4}\right) + \\
& A_{11}A_{13}U \frac{2at}{t+1}\left(-576D + 2.57143464 - 0.94285944 \frac{1}{t}\right) + A_{11}A_{31}U \frac{2at}{t+1}\left(-\frac{6912}{t} E + 768tN - 2.32002560 - 0.34228224 \frac{1}{t}\right) + \\
& A_{11}A_{33}U \frac{2at}{t+1}\left(-\frac{6912}{t} E + 2.09204160 - 1.19231904 \frac{1}{t}\right) + A_{13}A_{31}U \frac{2at}{t+1}\left(-\frac{6912}{t} E + 2.28251840 - 1.15422496 \frac{1}{t}\right) + \\
& A_{13}A_{33}U \frac{2at}{t+1}\left(-\frac{6912}{t} E + 2.380536 - 0.29949696 \frac{1}{t} - 0.49552128 \frac{1}{t^2}\right) + \\
& A_{31}A_{33}U \frac{2at}{t+1}\left(-\frac{82944}{t^2} F + 1.49446272 - 1.49127168 \frac{1}{t} - 0.26459236 \frac{1}{t^2}\right) + \\
& A_{11}A_{13}A_{31}\left(\frac{27648}{t^2} F + 3.911072 - 0.40761344 \frac{1}{t} + 0.08819712 \frac{1}{t^2}\right) + \\
& A_{11}A_{13}A_{33}\left(\frac{27648}{t^2} F + 4.10292624 + 0.53652480 \frac{1}{t} - 0.35664768 \frac{1}{t^2}\right) + \\
& A_{11}A_{31}A_{33}\left(\frac{331776}{t^3} G + 4.31941632 - 0.07341312 \frac{1}{t} + 0.22560768 \frac{1}{t^2} + 0.10285056 \frac{1}{t^3}\right) + \\
& A_{13}A_{31}A_{33}\left(\frac{331776}{t^3} G + 4.77937536 + 0.95553024 \frac{1}{t} - 0.18528960 \frac{1}{t^2} + 0.10285056 \frac{1}{t^3}\right)
\end{aligned}
$$

$$K_4 = A_{11}^3\left(\frac{384}{t} E + 0.49171200 + 0.13217280 \frac{1}{t}\right) +$$

$$A_{13}^3\left(\frac{384}{t} E + 1.42932978 + 0.55309932 \frac{1}{t} - 0.13175400 \frac{1}{t^2} - 0.14514480 \frac{1}{t^3} - 0.02992983 \frac{1}{t^4}\right) +$$

$$A_{31}^3\left(\frac{663552}{t^4} H + 0.61255680 + 0.08412672 \frac{1}{t} + 0.08626176 \frac{1}{t^2} + 0.04644864 \frac{1}{t^3} + 0.01327104 \frac{1}{t^4}\right) +$$

$$A_{33}^3\left(\frac{663552}{t^4} H + 1.33485408 + 0.64514880 \frac{1}{t} - 0.04784184 \frac{1}{t^2} - 0.09217152 \frac{1}{t^3} - 0.01550664 \frac{1}{t^4}\right) +$$

$$A_{11}^2 U \frac{2at}{t + 1}\left(-288D + 1.28571732 - 0.47142972 \frac{1}{t}\right) + A_{11}^2 A_{13}\left(\frac{1152}{t} E + 1.89610200 + 0.13562784 \frac{1}{t} - 0.23884706 \frac{1}{t^2}\right) +$$

$$A_{11}^2 A_{31}\left(\frac{13824}{t^2} F + 1.80954432 - 0.23556096 \frac{1}{t} + 0.04409856 \frac{1}{t^2}\right) +$$

$$A_{11}^2 A_{33}\left(\frac{13824}{t^2} F + 1.91459592 + 0.22063104 \frac{1}{t} - 0.18427392 \frac{1}{t^2}\right) +$$

$$A_{13}^2 U \cdot \frac{2at}{t + 1}\left(-288D + 2.18750217 + 0.48905445 \frac{1}{t} - 0.39710994 \frac{1}{t^2} - 0.15087705 \frac{1}{t^3}\right) +$$

$$A_{13}^2 A_{11}\left(\frac{1152}{t} E + 2.72042070 + 0.69054222 \frac{1}{t} - 0.34588152 \frac{1}{t^2} - 0.14399928 \frac{1}{t^3}\right) +$$

$$A_{13}^2 A_{31}\left(\frac{13824}{t^2} F + 2.90765532 + 0.85941696 \frac{1}{t} - 0.26121856 \frac{1}{t^2} - 0.13808882 \frac{1}{t^3}\right) +$$

$$A_{13}^2 A_{33}\left(\frac{13824}{t^2} F + 4.43507589 + 1.84144320 \frac{1}{t} - 0.28275813 \frac{1}{t^2} - 0.41674986 \frac{1}{t^3} - 0.08774370 \frac{1}{t^4}\right) +$$

$$A_{31}^2 U \frac{2at}{t + 1}\left(-\frac{41472}{t^2} F + 0.74723136 - 0.74563584 \frac{1}{t} - 0.13229618 \frac{1}{t^2}\right) +$$

$$A_{31}^2 A_{11}\left(\frac{165888}{t^3} G + 2.15970816 - 0.03670656 \frac{1}{t} + 0.11280384 \frac{1}{t^2} + 0.05142528 \frac{1}{t^3}\right) +$$

$$A_{31}^2 A_{13}\left(\frac{165888}{t^3} G + 2.47248768 + 0.50808576 \frac{1}{t} + 0.08864160 \frac{1}{t^2} + 0.05142528 \frac{1}{t^3}\right) +$$

$$A_{31}^2 A_{33}\left(\frac{1990656}{t^4} H - 2.91644928 + 0.97228800 \frac{1}{t} + 0.06590592 \frac{1}{t^2} + 0.13934592 \frac{1}{t^3} + 0.03981312 \frac{1}{t^4}\right) +$$

$$A_{33}^2 U \frac{2at}{t + 1}\left(-\frac{41472}{t^2} F + 1.53456444 + 0.16303680 \frac{1}{t} - 0.48873456 \frac{1}{t^2} - 0.14364378 \frac{1}{t^3}\right) +$$

$$A_{33}^2 A_{11}\left(\frac{165888}{t^3} G + 2.88453312 + 0.93080016 \frac{1}{t} - 0.18912528 \frac{1}{t^2} - 0.08430480 \frac{1}{t^3}\right) +$$

$$A_{33}^2 A_{13}\left(\frac{165888}{t^3} G + 4.58072496 + 2.01231216 \frac{1}{t} - 0.16849134 \frac{1}{t^2} - 0.35067168 \frac{1}{t^3} - 0.08593452 \frac{1}{t^4}\right) +$$

$$A_{33}^2 A_{31}\left(\frac{1990656}{t^4} H + 3.54741120 + 1.50133824 \frac{1}{t} + 0.00316224 \frac{1}{t^2} + 0.01016064 \frac{1}{t^3} + 0.03981312 \frac{1}{t^4}\right) +$$

$$A_{11} A_{13} U \frac{2at}{t + 1}\left(-576D + 2.87500632 - 0.01465512 \frac{1}{t} - 0.51822222 \frac{1}{t^2}\right) +$$

$$A_{11} A_{31} U \frac{2at}{t + 1}\left(-\frac{6912}{t} E + 2.09204160 - 1.19231904 \frac{1}{t}\right) +$$

$$A_{11} A_{33} U \frac{2at}{t + 1}\left(-\frac{6912}{t} E + 2.380536 - 0.29949696 \frac{1}{t} - 0.49552128 \frac{1}{t^2}\right) +$$

$$A_{13} A_{31} U \frac{2at}{t + 1}\left(-\frac{6912}{t} E + 2.380536 - 0.29949696 \frac{1}{t} - 0.49552128 \frac{1}{t^2}\right) +$$

$$A_{13} A_{33} U \frac{2at}{t + 1}\left(-\frac{6912}{t} E + 3.77748180 + 0.65672488 \frac{1}{t} - 0.75665232 \frac{1}{t^2} - 0.29417024 \frac{1}{t^3}\right) +$$

$$A_{31} A_{33} U \frac{2at}{t + 1}\left(-\frac{82944}{t^2} F + 1.75115664 - 0.60931584 \frac{1}{t} - 0.73367424 \frac{1}{t^2}\right) +$$

$$A_{11} A_{13} A_{31}\left(\frac{27648}{t^2} F + 4.10292624 + 0.53652480 \frac{1}{t} - 0.35664768 \frac{1}{t^2}\right) +$$

$$A_{11} A_{13} A_{33}\left(\frac{27648}{t^2} F + 5.43208248 + 1.55212416 \frac{1}{t} - 0.55813536 \frac{1}{t^2} - 0.27949212 \frac{1}{t^3}\right) +$$

$$A_{11} A_{31} A_{33}\left(\frac{331776}{t^3} G + 4.50337536 + 0.85446144 \frac{1}{t} - 0.19863360 \frac{1}{t^2} + 0.10285056 \frac{1}{t^3}\right) +$$

$$A_{13} A_{31} A_{33}\left(\frac{331776}{t^3} G + 5.88498624 + 1.91466144 \frac{1}{t} - 0.36624096 \frac{1}{t^2} - 0.16747776 \frac{1}{t^3}\right)$$

The definite integrals A, B, C, and so forth that occur in the foregoing equations are:

$$A = \int_1^\infty \frac{(y-1)^4\,dy}{y^3(y+1)^2(yt-1)}$$

$$B = \int_1^\infty \frac{(y-1)^4\,dy}{y^2(y+1)^4(yt-1)}$$

$$C = \int_1^\infty \frac{(y-1)\,dy}{(y+1)^6(yt-1)}$$

$$D = \int_1^\infty \frac{(y-1)^4\,dy}{(y+1)^8(yt-1)}$$

$$E = \int_1^\infty \frac{(y-1)^4\,dy}{(y+1)^{10}(yt-1)}$$

$$F = \int_1^\infty \frac{(y-1)^4\,dy}{(y+1)^{12}(yt-1)}$$

$$G = \int_1^\infty \frac{(y-1)^4\,dy}{(y+1)^{14}(yt-1)}$$

$$H = \int_1^\infty \frac{(y-1)^4\,dy}{(y+1)^{16}(yt-1)}$$

$$J = \int_1^\infty \frac{(2y+1)\,dy}{y^2(yt+1)}$$

$$K = \int_1^\infty \frac{dy}{y(yt+1)}$$

$$L = \int_1^\infty \frac{dy}{(y+1)^2(yt+1)}$$

$$M = \int_1^\infty \frac{y\,dy}{(y+1)^4(yt+1)}$$

$$N = \int_1^\infty \frac{y^2\,dy}{(y+1)^6(yt+1)}$$

$$P = \int_1^\infty \frac{y^3\,dy}{(y+1)^8(yt+1)}$$

$$Q = \int_1^\infty \frac{y^4\,dy}{(y+1)^{10}(yt+1)}$$

$$R = \int_1^\infty \frac{y^5\,dy}{(y+1)^{12}(yt+1)}$$

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TABLE 1

FLOW PAST ELLIPTIC CYLINDER WITH $b/a = 0.5$ AND $M_o = 0.5$

Eccentric angle, η (deg)	Velocity distribution, q/U			
	Incompressible flow	Variational method	Hooker's method	Kaplan's results (reference 2)
0	0	------	-----	------
15	.7085	0.6655	0.667	0.6683
30	1.1339	1.1054	1.130	1.1310
45	1.3417	1.3731	1.380	1.3869
60	1.4412	1.5450	1.513	1.5184
75	1.4867	1.6468	1.575	1.5809
90	1.5	1.6810	1.607	1.5994

TABLE 2

FLOW PAST AN ELLIPTIC CYLINDER WITH $b/a = 0.1$ AND $M_o = 0.8$

Eccentric angle, η (deg)	Velocity distribution, q/U			
	Incompressible flow	Variational method	Perl's solution (reference 4)	Kaplan's results (reference 2)
0	0	0	0	0
5	.7244	.6982	------	.6898
10	.9569	.9256	------	.9137
15	1.0307	1.0026	------	.9886
20	1.0608	1.0399	------	1.0237
30	1.0839	1.0845	1.145	1.0629
40	1.0924	1.1199	1.1625	1.0918
50	1.0962	1.1526	1.1754	1.1179
60	1.0983	1.1819	1.1827	1.1408
70	1.0994	1.2052	1.1883	1.1589
80	1.0999	1.2203	1.1903	1.1706
90	1.10	1.2255	1.1913	1.1747

TABLE 3

FLOW PAST A CIRCULAR CYLINDER WHEN $M_0 = 0.4$

Eccentric angle, η (deg)	Velocity distribution, q/U			
	Incompressible flow	Present solution with four terms	Wang's results (reference 9)	Rayleigh-Janzen method (third approximation)
0	0	------	------	0
10	.3473	0.2966	0.3194	.3104
20	.6840	.6002	.6349	.6439
30	1.0000	.9132	.9410	.9588
40	1.2856	1.2305	1.2470	1.2658
50	1.5321	1.5381	1.5522	1.5613
60	1.7321	1.8149	1.8450	1.8355
70	1.8794	2.0366	2.0974	2.0752
80	1.9696	2.1804	2.2712	2.2269
90	2.0000	2.2303	2.3335	2.2840

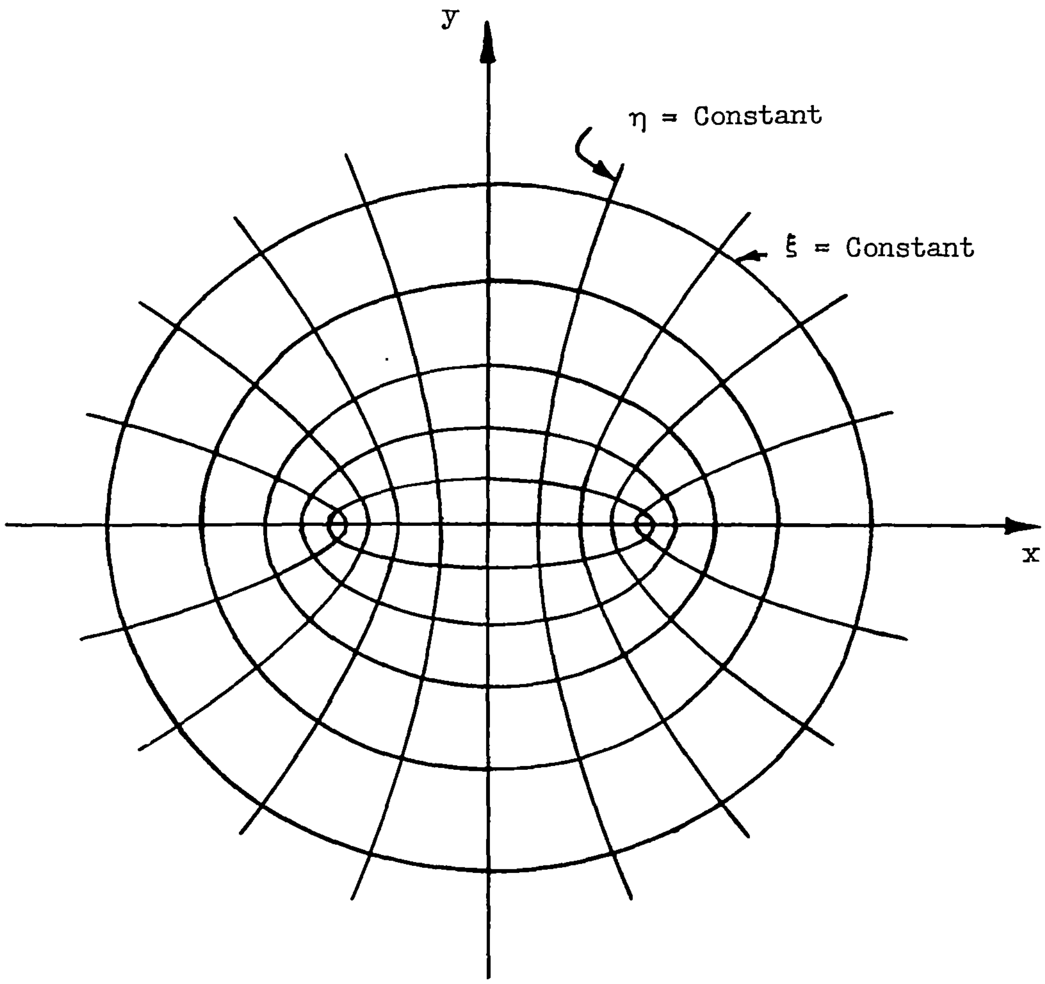

Figure 1.- Elliptic coordinates.

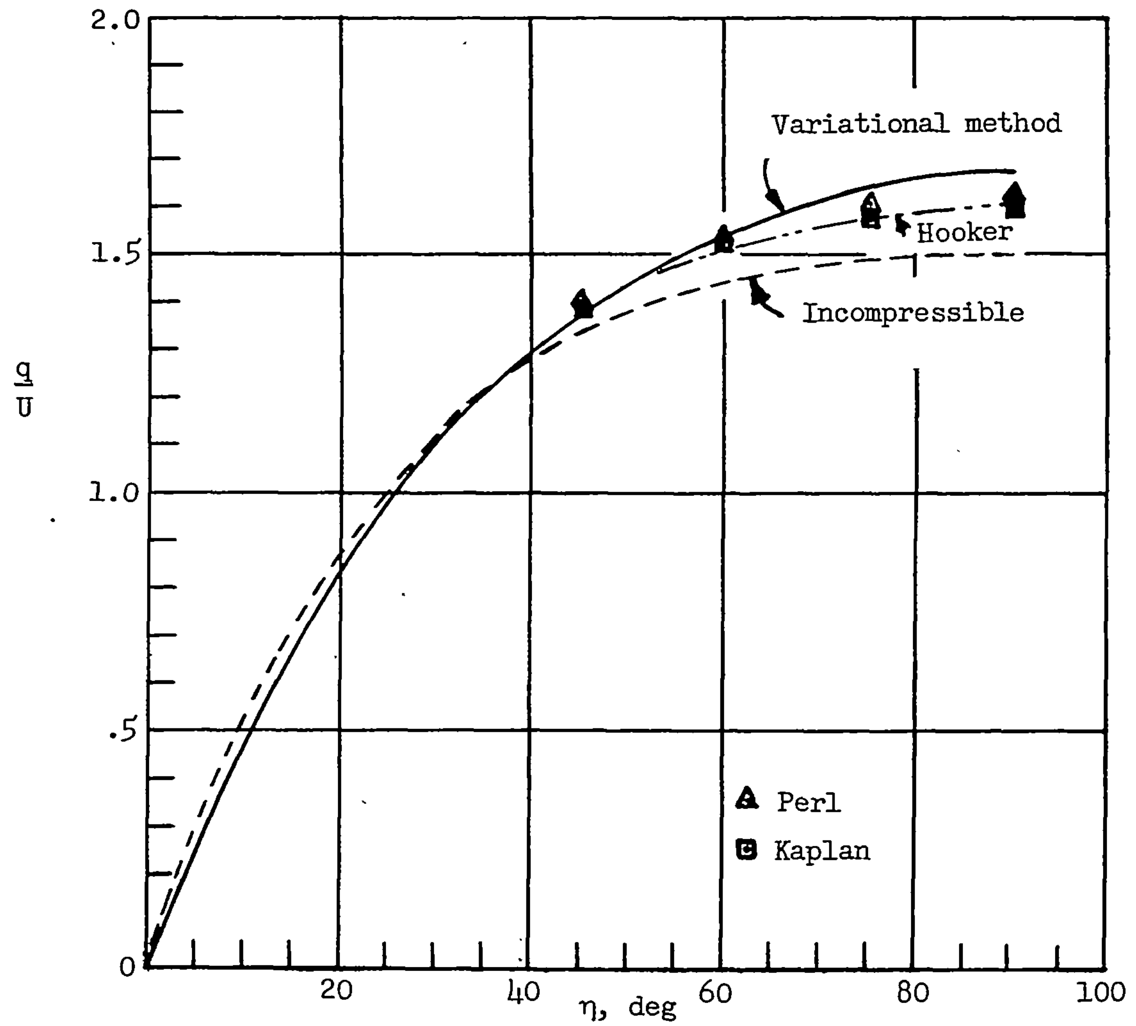

Figure 2.- Velocity distribution on surface of an elliptic cylinder of thickness ratio 0.5 when $M_o = 0.5$.

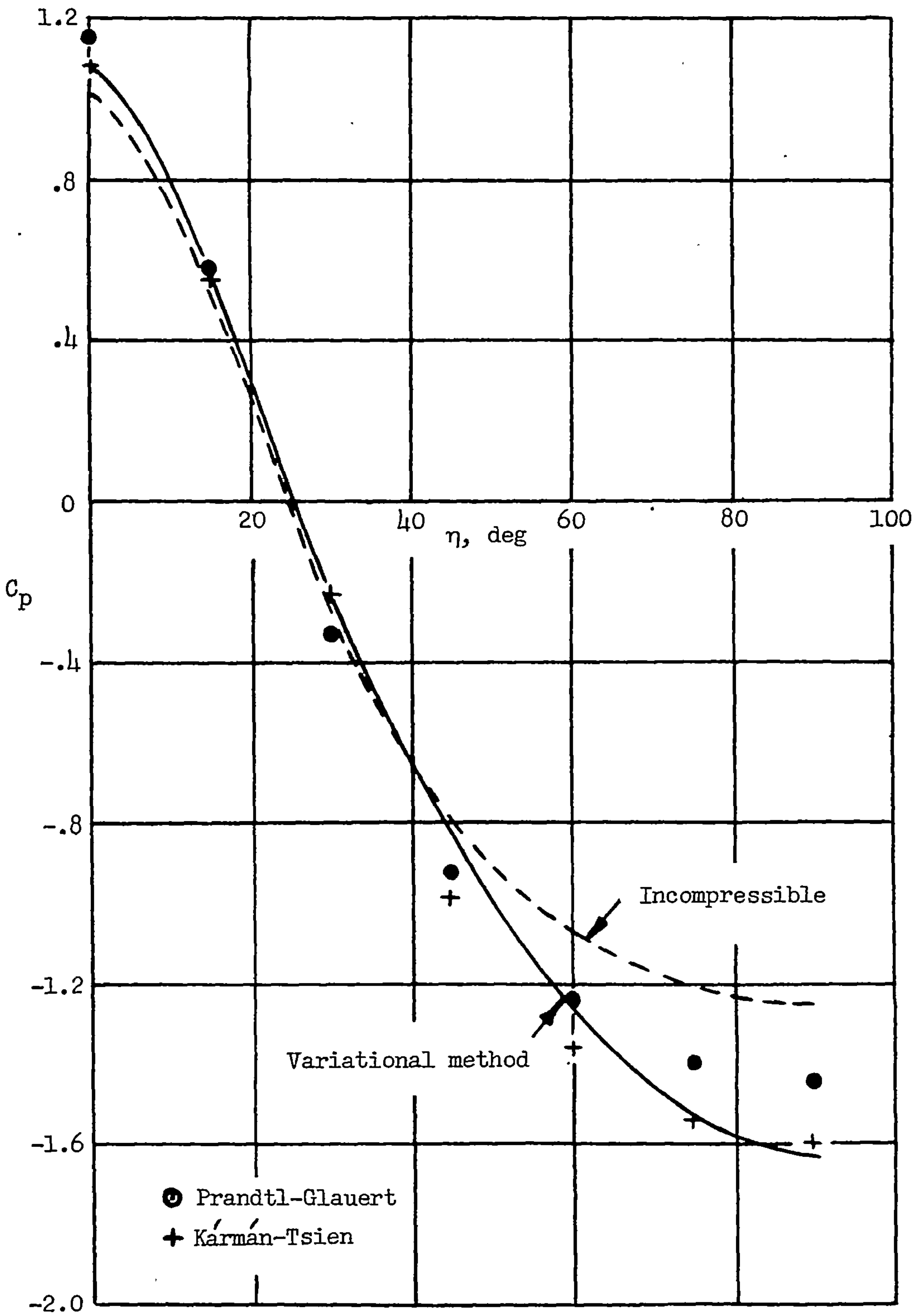

Figure 3.- Pressure distribution on surface of elliptic cylinder of thickness ratio 0.5 with $M_0 = 0.5$.

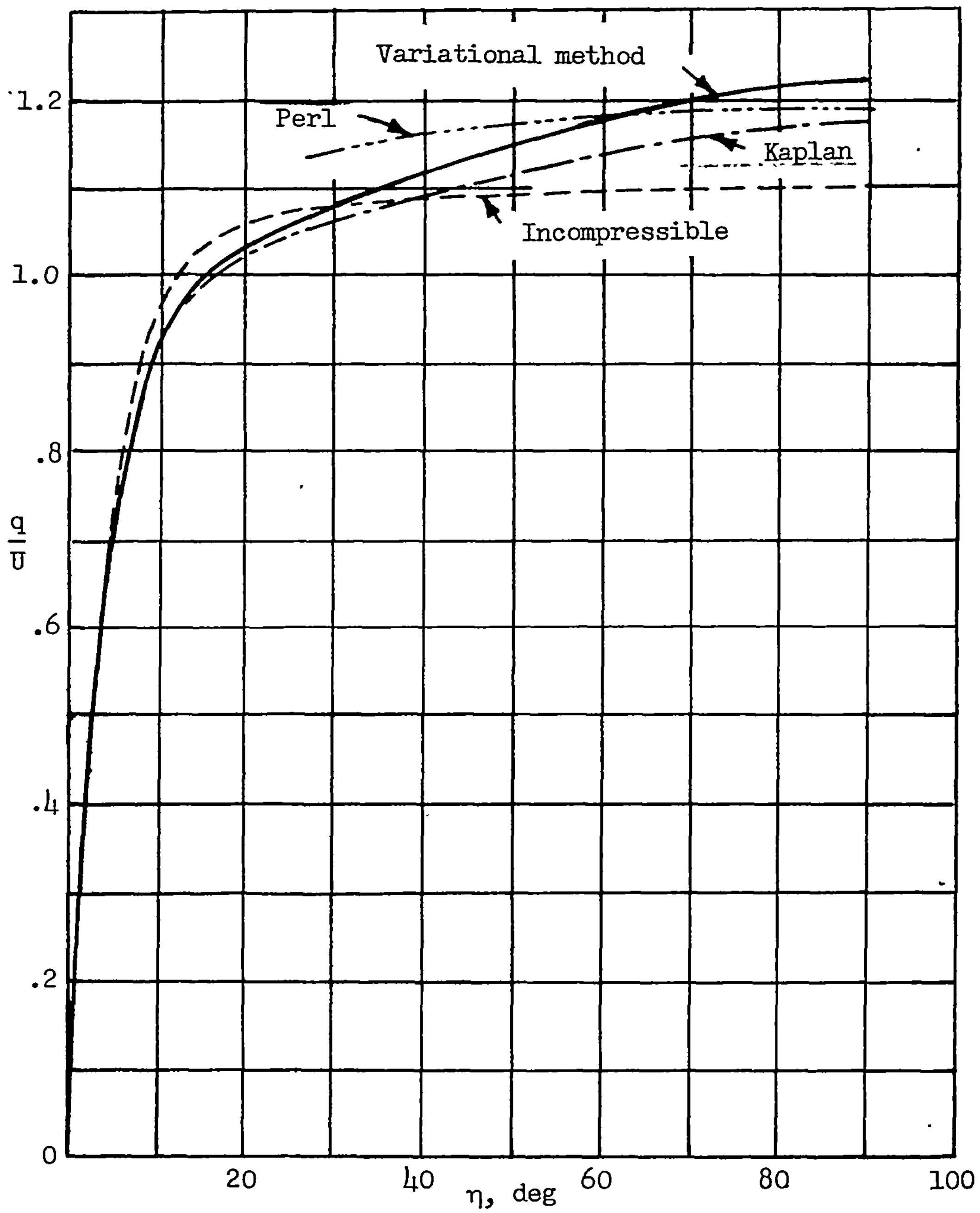

Figure 4.- Velocity distribution on surface of an elliptic cylinder of thickness ratio 0.1 when $M_0 = 0.8$.

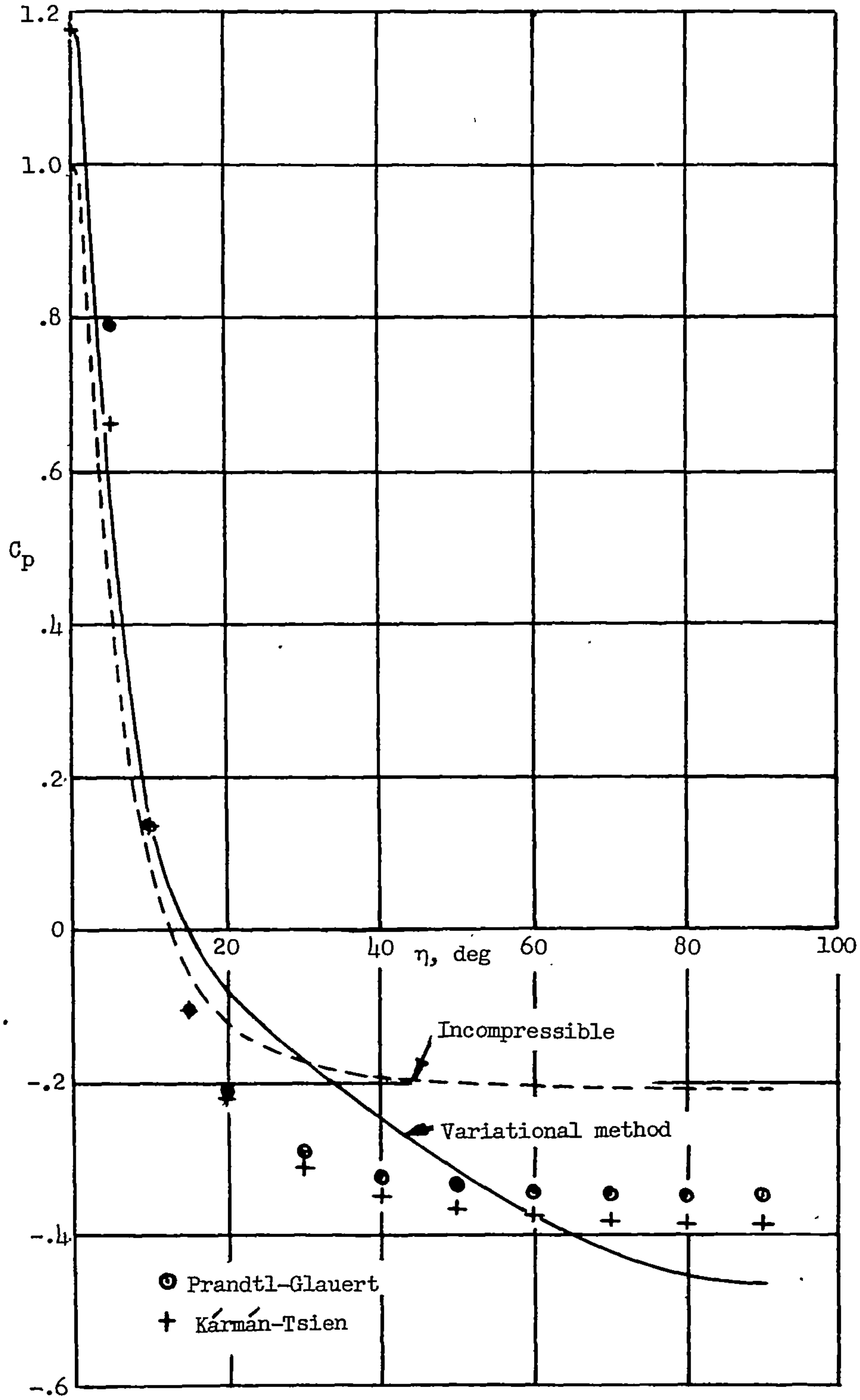

Figure 5.- Pressure distribution on surface of elliptic cylinder of thickness ratio 0.1 with $M_o = 0.8$.

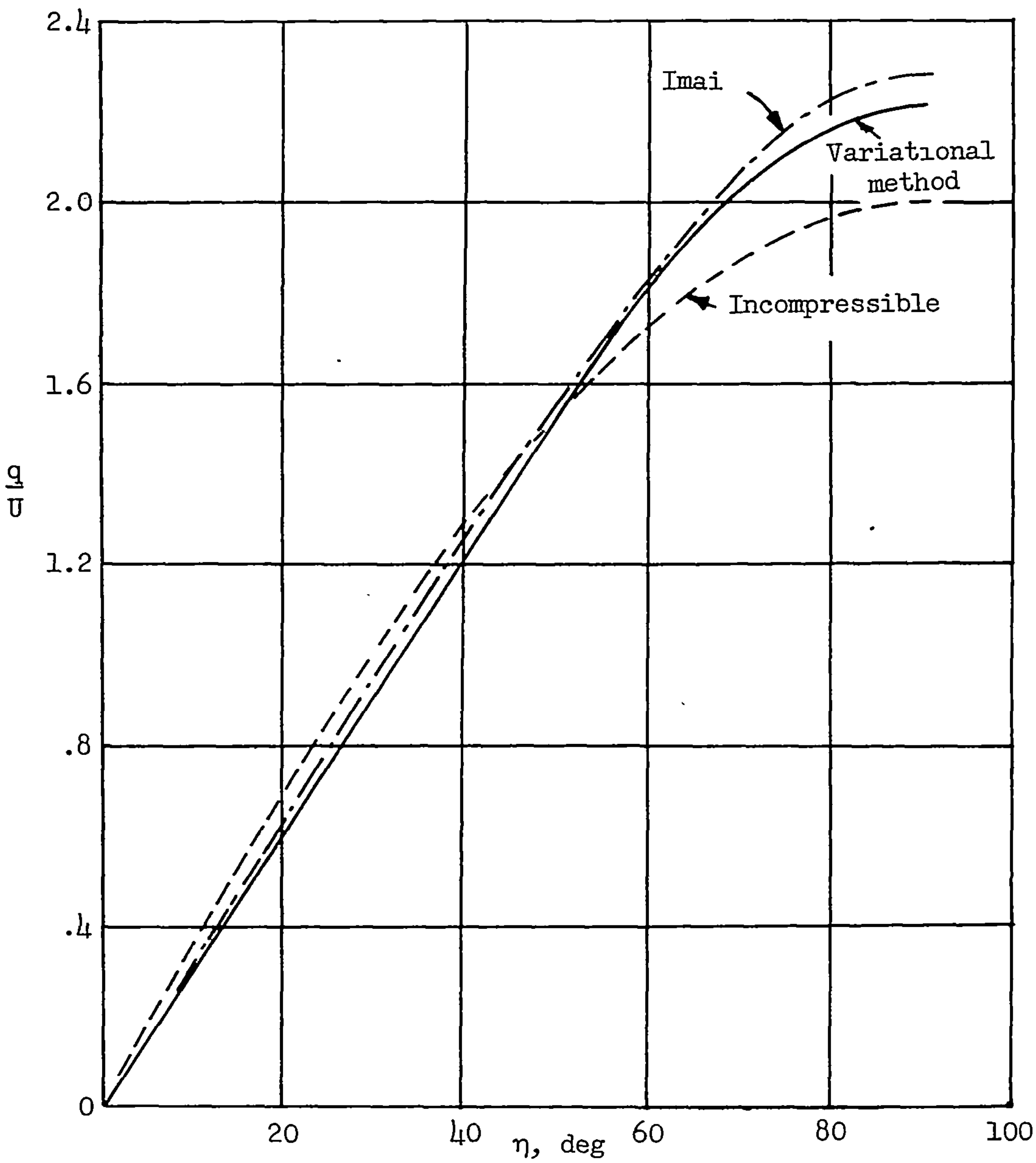

Figure 6.- Flow past a circular cylinder obtained by making t approach ∞, when $M_0 = 0.4$.

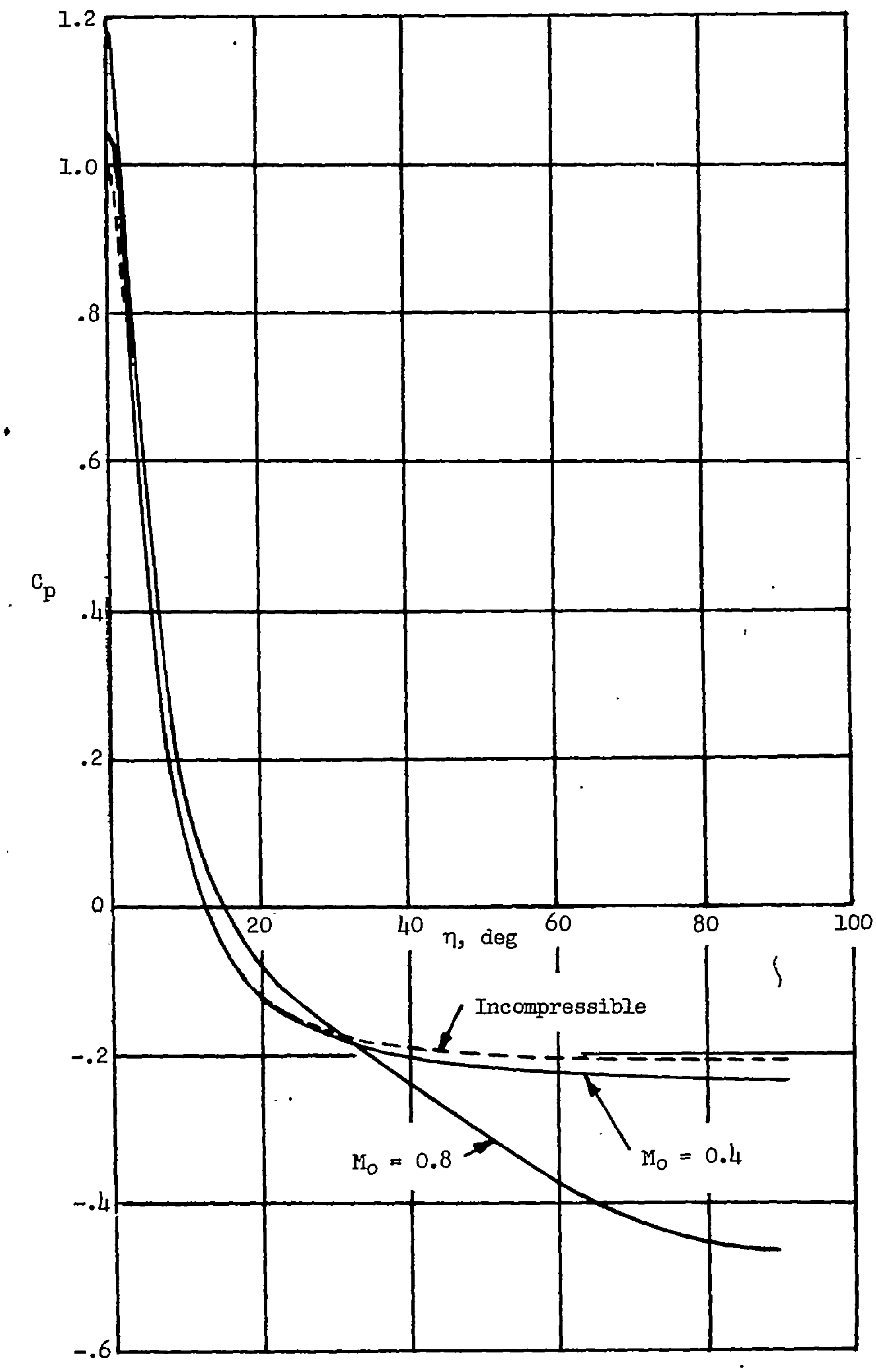

Figure 7.- Pressure distribution on surface of elliptic cylinder of thickness ratio 0.1 when $M_o = 0.4$ and 0.8.